Remarks on scaling properties inherent to the systems with hierarchically organized supplying network

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Abstract

We study the emergence of a power law distribution in the systems which can be characterized by a hierarchically organized supplying network. It is shown that conservation laws on the branches of the network can, at some approximation, impose power law properties on the systems. Some simple examples taken from economics, biophysics etc. are considered.

The existence of power law distribution in different systems of nature is a fascinating property still waiting for sufficient explanation. Despite a very long history of investigation of different scaling laws in systems of outworld this problem is very attractive up to the present time. A high interest to this problem is associated primarily with the fact that the systems, different in their nature, possess the similar power law distributions. This in its turn indicates that complex systems which consist of tremendous amount of objects have universal behavior, whose investigation gives the possibility to penetrate deeply into the essence of organization of nature.

V.Pareto (1897, [1]) seems to be first who discussed the power law distributions in wealth of the individuals in a stable economy. Since that time more than one century passed and the scaling properties are already discovered in many systems. Now we meet power law distributions in turbulence [2, 3], in biological systems [4], economics and finance [6, 5, 7, 8], social phenomena [9], linguistic phenomena [10], systems which can be characterized by network organization like river network [11, 12], etc.

In this paper we single out a certain subclass of systems widely met in the nature which obey power law. Main characteristic feature of such systems is their hierarchical organization. The hierarchical organization allows such system to be divided into sets of subsystems (which are called levels) involving many elements that are similar in their properties. The elements of the different levels

are substantially different in their characteristics.

We consider systems that can exist only due to the permanent flow through the system of some transport agents joining different subsystems and elements into a whole organization. Such interconnection of systems elements is formed by hierarchically organized supplying network. The specific function of the network is to drain the systems or supply them with some transport agents needed for systems activity. The characteristic feature of such system consists of conservation laws which should be fulfilled on each hierarchy level. This denotes that total flow of the transport agents through the system is constant.

We represent the system under consideration by a simple model schematically drawn on Fig. (1,a). Each hierarchy level i consists of set of n_i independent elements. All these elements take part in transferring transport agent flow from one level to another, namely taking it from the elements belonging to the lower level, and transferring it to the elements of higher level and

$$n_{i+1} > n_i > n_{i-1}; \quad \forall i.$$

Due to the transport agent flow being constant through each hierarchy level i the total flow X_{total} is constant and

$$X_{\text{total}} = \sum_j X_{i,j} \approx n_i X_i, \quad \forall i, \tag{1}$$

here n_i-is the number of system elements on the i-th hierarchy level, $X_{i,j}$-the flow through the j-th element of the i-th hierarchy level, X_i-the average over level flow through each element of level i. It was supposed that all the elements of the same hierarchy have similar characteristics and it is possible to sum up their flows directly. In the framework of this article we pay attention to the fact that the power law dependence is inherent to different natural systems and their occurrence is naturally connected to the function of systems and is the intrinsic property of the systems.

The flow X_i though each element is determined by characteristics of the properties of the element and system organization. We denote all these characteristics by some parameters A_i. Namely these parameters are observable features of systems under consideration. It is conventional way to investigate power law distribution of these parameters. In this case from (1) we get an obvious relation

$$n_i \backsim 1/X_i(A_i). \tag{2}$$

We will touch some examples of model to determine A_i.

Now consider a simple example taken from economics. As the economic systems exist only with the streams of commodity and money, they possess all the above-listed properties.

The number of parameters, which define the economic market state, at first glance seems to be actually endless because there is a tremendous amount of goods on the market. At the same time a relatively small number of raw materials in the industry shows that there should be a large hierarchical systems

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in the market structure. So it is possible to present an economic system as a hierarchical system which hierarchy levels consist of the firms of different power. On the first level there are firms processing row materials, on the last level – the shops of retail trade. In this context we point out that after reaching the consumers goods the flows transform into money flows are going in the opposite direction. The schematic representation of such system is shown in fig. (1,b). We shall consider the situations, when the market of the considered goods exist, i.e. all the streams should be more than zero.

For simplicity we confine our consideration to some commodity market, in which producers, dealers and consumers of some kinds of goods will form a common connected network. For example, the market of steel or meat items and the market of furniture can interact with each other mainly through the changes in financial states of the whole set of the consumers. In this case we can consider some branch of industry as a single one which practically does not interact with any other branches of industry [13, 14, 15, 16]. [1]

The level i of this system is a set of n_i independent firms, bringing out the products of some sort, which buy the products from firms of the lower level of the hierarchy, and sell the results of their activity to the firms of the higher level. Firms, forming the common level i, are considered as identical from the viewpoint of their power. The bottom of the system (the firms of the first level) is formed by the firms, obtaining and processing raw material. The market participants of the last level are the "points" of retail trade, supplying the consumers with required goods.

The level of the firm production on each level i is measured in the units of the initial raw material flow X_i, "flowing" through the given firm. It should be noted, that in this system of units the value X having dimensionality (*material*)/(*time*). Particular interconnection between different firms can occur and disappear during the formation and evolution of the market under consideration. Their interaction is governed by trade with firms belonging to the neighbor levels.

For simplicity, consider some hierarchical system which can be represented as a homogeneous tree. Each firm is represented by the node on the tree. One branch enter this node and a branches go out. Such representation denotes, that the given firm connected with trade relations with one firm of the lower level and a firms of the higher level. It is assumed that tree consist of N levels. In this case number of firms on i-th level - n_i - equal

$$n_i = a^{i-1}. \tag{3}$$

The fraction of the system, which make up the i-th level is

[1]It should be noted that this restriction is not so strong. Main conclusion of this article will be true for the whole economic system (for example, the economic system of the state), including many sets of material flows twisted with each other. Indeed, in this case we can expend total material flow onto the basis of different material flows. For each of them situation will be practically the same as listed below.

Figure 1: Representation of the flows between levels in the systems with hierarchically organized supplying structure. In Fig. a: segments correspond to firms and arrows to flows; the volume of the segments and arrows represent hierarchical organization of the system; Fig. b corresponds to the schematic representation of market structure in the tree form.

$$p(i) = \frac{n_i}{S_N}, \tag{4}$$

where S_N - number of nodes in the tree (firms in our system)

$$S_N = \sum_{i=1}^{N} n_i = \frac{a^N - 1}{a - 1}. \tag{5}$$

The fraction $p(i)$ may be regarded as the probability that chosen at random firm belong to the i-th level. Cumulative probability defined as the probability that chosen at random firm belong to one of levels from the 1-st to the i-th

$$P(i) = \frac{S_i}{S_N} = \frac{an_i - 1}{a^N - 1}, \tag{6}$$

where $S_i = \sum_{k=1}^{i} n_k$.

As it was mentioned above $n_i = X_{\text{total}}/X_i$ due to this fact probability to choose node with flow x less or equal to X_i (cumulative distribution function) is

$$P(x \geqslant X_i) = \frac{a(X_{\text{total}}/X_i) - 1}{a^N - 1} \backsim \frac{1}{X_i}. \tag{7}$$

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Finally, it is important to express flow X_i using the dependence of the parameter A_i from the flow X_i. There is a question what the parameter A_i means. For example let us consider the parameter A_i as a firm's profit π_i. It is natural to suppose that profit of each firm is proportional to the material or financial flow through the firm, namely $\pi_i \backsim X_i^{\alpha}$ ($\alpha > 0$). Then we can write cumulative probability distribution of firm's profit for $n_i \gg 1$

$$P(\geqslant \pi_i) \backsim \frac{1}{\pi_i^{1/\alpha}}. \tag{8}$$

In the most intuitive case $\alpha = 1$ and we obtain Zipf's law distribution as it was observed in [5]. It seems to be reasonable to note that distribution of the firms from their income analyzed in the [5] is a direct consequence of the hierarchical organization of the firms into corresponding industrial branches, which are connected to each other by material and financial flows.

As an another typical example of the systems under consideration we may regard living tissue where blood, flowing through the vascular network, which involves arterial and venous beds, supplies cellular tissue with oxygen, nutritious products, etc. At the same time blood withdraws carbon dioxide and products resulting from life activities of the cellular tissue. Both the arterial and venous beds being of the tree form contain a large number of hierarchy levels and are similar in structure. A similar situation takes place in respiratory systems where oxygen going through hierarchical system of bronchial tubes reaches small vessels (capillaries).

A microcirculatory bed of living tissue can be reasonably regarded as a space-filling fractal, being a natural structure for ensuring that all cells are serviced by capillaries [17, 18]. The vessel network must branch so that every small group of cells, referred below to as "elementary tissue domain", is supplied by at least one capillary. In this case a vessel network generated by arteries should contain a sufficiently large number of hierarchy levels At each level i of the vascular network the tissue domain supplied by a given microcirculatory bed as a whole can be approximated by the union of the tissue subdomains whose mean size is about the typical length l_i of the i-th level vessels. Thus, the individual volume of these subdomains is estimated as $V_i \sim l_i^3$. From this simple consideration immediately follows the sufficiently obvious power law distribution for vascular network. Indeed, for the space-filling vascular network this artery supplies with blood the tissue region of volume about l_i^3 and, so, under normal conditions the blood flow rate X_i in it should be equal to

$$X_i \approx j l_i^3, \tag{9}$$

where j is the blood perfusion rate (the volume of blood, flowing through the tissue domain of volume unit per time unit) assumed to be the same at all the points of the given microcirculatory bed. In living tissue the ratio l_i/a_i takes a certain fixed value, $l_i \approx \text{constant} \cdot a_i$ and, thus, $X_i \approx \text{constant}' \cdot a_i^3$. Due to the blood conservation at branching nodes the scaling law of such system of vessels

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controlling finally the blood flow redistribution over the microcirculatory beds can be represented as

$$n_i \backsim const/a_i^3 \backsim const/l_i^3 \qquad (10)$$

In the same way we can obtain approximate power law dependance for river network organization. Recapitulating practically one-to-one the explanation listed above, for the hierarchically river network as two-dimensional space-filling fractal we can write

$$X_i \approx jl_i^2.$$

In this case from equation (1) we get

$$n_i \backsim const/l_i^2 \backsim const/l_i^2. \qquad (11)$$

The obtained power law dependence is approximately valid for qualitative understanding of scaling properties so as the results obtained in framework of this approach correspond to that one, obtained by more rigorous consideration [11].

From the described examples we can develop a general mathematical model of the system under consideration consisting from the distributed basic medium $\mathbb{M}$ and the transport hierarchical network S (Fig. 1,b). The transport network can supply basic medium by some transport agent or can drain basic medium. The medium $\mathbb{M}$ is a d - dimensional homogeneous continuum. The transport network can be reasonably regarded as a space-filling fractal embedded in d-dimensional homogeneous continuum.

$$X_i \approx jl_i^d$$

where j is the flow of transport agents (the volume of transport agent through some domain of unit volume per unit time) assumed to be the same at all the points of the given bed. In this case $n_i \approx const/l_i^d$, and the dimensionality of homogeneous continuum should be determined. In the given examples such basic medium has dimensions $d = 1$, $d = 3$ and $d = 2$ respectively to (9),(10),(11). In general case such dimensions can be noninteger and their properties seem to determine the properties of basic medium $\mathbb{M}$ as a whole.

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